Announcements

1) External Reviewers

Would like to meet with math majors from 2:15 - 3 Tuesday 3/12 in CB 2047 (Math Library). There will be cookies!

2) New Webwork up later today

Example from last time

<u>Note</u>: Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\{b_i\}_{i=1}^n$ is a basis for $\mathbb{R}^n$. Let $\{f_i\}_{i=1}^n$ be another basis for $\mathbb{R}^n$ and let S be the (invertible) $n \times n$ matrix that solves

$$S \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If the matrix of T with respect to $\{b_i\}_{i=1}^n$ is A, then the matrix of T with respect to $\{f_i\}_{i=1}^n$ is

$$\boxed{SAS^{-1}}$$

Example 1:

Let $\{b_1, b_2, b_3\}$ be a basis

$$b_1 = \begin{bmatrix} -6 \\ 1 \\ 4 \end{bmatrix}$$

$$b_2 = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$b_3 = \begin{bmatrix} -1 \\ 9 \\ 0 \end{bmatrix}$$

of $\mathbb{R}^3$.

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T(b_1) = -b_2 + 3b_3$$

$$T(b_2) = 6b_1 - 4b_2 + 2b_3$$

$$T(b_3) = 22b_1.$$

Find

a) the matrix of T with respect to $\{b_1, b_2, b_3\}$

b) the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

a) In column form,

$$T(b_1) = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$T(b_2) = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix}$$

$$T(b_3) = \begin{bmatrix} 22 \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

we get

$$\begin{bmatrix} 0 & 6 & 22 \\ -1 & -4 & 0 \\ 3 & 2 & 0 \end{bmatrix} (= A)$$

b) $S = \begin{bmatrix} -6 & 5 & -1 \\ 1 & 8 & 9 \\ 4 & 3 & 0 \end{bmatrix}$

$\det(S) = 371 \neq 0,$

so S is invertible.

$$S^{-1} = \frac{1}{371}\begin{bmatrix} -27 & -3 & 53 \\ 36 & 4 & 53 \\ -29 & 38 & -53 \end{bmatrix}$$

(found using Wolfram Alpha)

The matrix we're looking for is

$$SAS^{-1} =$$

$$\underbrace{\begin{bmatrix} -6 & 5 & -1 \\ 1 & 8 & 9 \\ 4 & 3 & 0 \end{bmatrix}}_{S} \cdot \underbrace{\begin{bmatrix} 0 & 6 & 22 \\ -1 & -4 & 0 \\ 3 & 2 & 0 \end{bmatrix}}_{A} \cdot \underbrace{\frac{1}{371}\begin{bmatrix} -27 & -3 & 53 \\ 36 & 4 & 53 \\ -29 & 38 & -53 \end{bmatrix}}_{S^{-1}}$$

Use Wolfram Alpha or your calculator to compute the final answer. We get

$$\frac{1}{371}\begin{bmatrix} 1956 & -5224 & 3498 \\ -1439 & 747 & -583 \\ -2039 & 3401 & -4187 \end{bmatrix}$$

Definition: (isomorphism)

Let V and W be two vector spaces. A linear transformation $T: V \to W$ is called an isomorphism if T is one-to-one and onto (the range of T is all of W). Then V and W are said to be isomorphic.

Theorem: (finite dimensional vector spaces)

Any finite-dimensional vector space is isomorphic to $\mathbb{R}^n$.

proof: (sketch) Let V have dimension $n < \infty$. Let $\{b_i\}_{i=1}^n$ be a basis for V.

Let $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$.

Define $T: V \to \mathbb{R}^n$,

$$T\left(\sum_{i=1}^{n} c_i b_i\right) = \sum_{i=1}^{n} c_i e_i$$

for any scalars $\{c_i\}_{i=1}^{n}$.

Check that T is one-to-one and onto. $\square$

Consequence: (linear transformations)

Any linear transformation between finite dimensional spaces may be expressed as a matrix (with respect to some basis).